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AREA AND THE LENGTH OF THE SHORTEST CLOSED GEODESIC

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1. Introduction

The main purpose of this paper is to prove the following theorem:

Theorem 4.2. *For any metric on a two-dimensional sphere* $31\sqrt{A} \geqslant L$, *where A represents the area, and L the length of the shortest nontrivial closed geodesic.*

The constant 31 above is not the best constant. One suspects that the best constant would be $3^{1/4}2^{1/2} \simeq 1.86121$. We will discuss this later.

The question, for which the above theorem is the answer in two dimensions, was posed by Gromov for all closed manifolds in [12, p. 135]. The corresponding theorem is known for all other closed surfaces as we will see below. The difficulty with the sphere is that it is simply connected. In fact, all other results relating area (or volume) to the length of closed geodesics concern essential (not null homotopic) geodesics. That is, they concern $\mathrm{sys}(M)$ (read "systole of M"), the length of the shortest essential geodesic.

The first theorem of this type was proved by Loewner in 1949 in an unpublished work (see [3] and [4]). He showed that for any metric on the two torus $3^{1/4}2^{1/2}\sqrt{A(M)} \geqslant \mathrm{sys}(M)$ with equality holding if and only if M is a flat equilateral torus. (The fact that the constant is the same as the conjectured constant for S^2 comes from the fact that both extremal metrics are built out of two flat equilateral triangles.) The proof of the theorem relies on the fact that all metrics are conformal to a flat metric. Using a similar method Pu in 1952 (see [17]) showed that for any metric on $\mathbf{RP}^2$; $\sqrt{\pi/2}\sqrt{A(M)} \geqslant \mathrm{sys}(M)$, with equality holding if and only if M has constant curvature.

In 1960 Accola [1] and Blatter [6] independently showed that there was a function $f(g)$ such that for any metric on a surface of genus g, $f(g)\sqrt{A(M)} \geqslant \mathrm{sys}(M)$. Unfortunately, as g tends to ∞ the function $f(g)$ tends to ∞ while one would expect it to tend to 0. In 1981 Hebda [15] and independently

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Burago and Zalgaller [7] improved this result by showing that for all surfaces of genus $\geqslant 1$, $\sqrt{2}\sqrt{A(M)} \geqslant \mathrm{sys}(M)$.

The major work in this field is the recent work of Gromov [12]. In it are many results. Among them is the result that for a surface of genus g, $\tilde{f}(g)\sqrt{A(M)} \geqslant \mathrm{sys}(M)$, where, in this case, $\tilde{f}(g)$ is a function (given explicitly) which tends to 0 as g tends to ∞ (see [12, pp. 4–5], Theorem 0.2.A). The main result of [12] is a higher dimensional theorem: There is a constant $c(n)$ depending only on dimension n such that for every essential manifold M of dimension n we have $c(n)\sqrt[n]{\mathrm{Vol}(M)} \geqslant \mathrm{sys}(M)$ ($c(n)$ can be taken to be $6(n+1)n\sqrt[n]{(n+1)!}$). In the above statement "M essential" means that there is a map f from M to a $k(\pi, 1)$ such that $f_*[M] \neq 0$ where $[M] \in H_n(M)$ is the fundamental class (use $\mathbf{Z}_2$ coefficients if M not orientable). As examples of esssential manifolds we have $\mathbf{RP}^n$ and $\mathbf{T}^n$.

It is easy to construct examples of nonessential manifolds with $\mathrm{sys}(M) = 1$ and arbitrarily small volume, for example take product metrics on $S^1 \times S^2$ where S^2 gets arbitrarily small. However, they may still have short nonessential closed geodesics. The general question of volume versus the length of the shortest closed geodesic is still very much open in higher dimensions. The goal of this paper is to answer the question in two dimensions.

The paper is divided into six sections, the first of which being this introduction. The second section recalls the Birkhoff curve shortening process, the fundamental tool in this paper, and derives some new properties. The third section contains the basic lemmas from which the theorems are proved in §§4 and 5. The key to all the proofs is Lemma 3.1.

In §4 the main theorem is proved. Along the way we also prove

Theorem 4.1. *For any metric on S^2 we have $9D \geqslant L$ where D represents the diameter.*

In §5 we consider complete noncompact surfaces of finite area A. It was shown in [20] and [2] that all such surfaces have closed geodesics (in fact infinitely many). Gromov in [12] showed (as a special case of Theorem 4.4.A) that for most such surfaces we have $\mathrm{const}\sqrt{A} \geqslant L$. We show that the techniques used to prove the main theorem serve to show $\mathrm{const}\sqrt{A} \geqslant L$ for all such surfaces. The two main cases not covered by Gromov's theorem are the plane and the cylinder.

In §6 we consider the case of convex hypersurfaces of $\mathbf{R}^{n+1}$. We show

Theorem 6.1. *If $M^n \subset \mathbf{R}^{n+1}$ is a convex hypersurface, then $c(n)\sqrt[n]{\mathrm{Vol}(M)} \geqslant L$.*

The constant $c(n)$ is discussed and is in some sense only off by a factor of two from the sharp constant. In particular $c(2) = 2 \cdot 3^{1/4} \cdot 2^{1/2}$. The argument

leads one to guess as to the optimal metric. In particular the extremal metric in two dimensions should be two copies of an equilateral triangle glued together along the boundary (of course this is a degenerate metric).

Andre Treibergs [19] has independently proved Theorem 6.1 as well as an extension to higher dimensional minimax volumes (yielding in particular upper bounds for areas of minimal surfaces in convex three spheres). We have nevertheless included our proof (which was proved at about the same time—Summer 1983) because it is significantly easier than the proof in [19] and it leads one to a conjecture as to the sharp constant. In [19] a different metric is conjectured as optimal. However, the geodesic used in the calculation in [19] was not the shortest closed geodesic. In fact the extremal metric conjectured above (two equilateral triangles) is better than the one suggested in [19].

Gromov has suggested that using the main Lemma (3.1) along with ideas in [12] one should be able to show that the filling radius (see [12] §1 for a definition) is larger than a constant times the length of the shortest closed geodesic. This, along with the main theorem (1.3.A) of [12], its extension ([12] 4.4.C), and results in [16], would yield an alternative proof of the main results in this paper.

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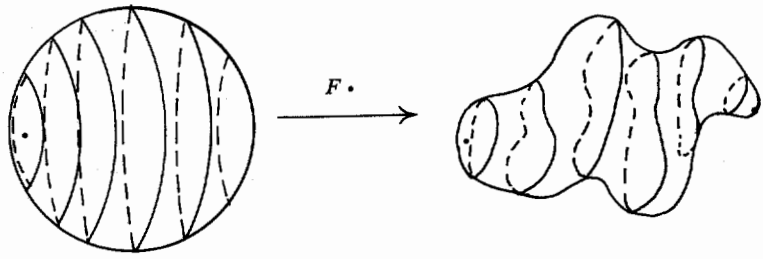
2. Birkhoff's Ideas

In this paper we will work in the space Λ of piecewise smooth closed curves $\gamma\colon [0,1] \to M$, where M is a riemannian manifold and Λ has the C^0-topology. By $L(\gamma)$ we will mean the length of γ.

We borrow two major ideas from Birkhoff (see [5]). The first idea we will use is his method of finding closed geodesics on spheres. In particular when M is S^2 we will find a 1-parameter family of curves starting and ending at a point curve in such a way that the induced map $F\colon S^2 \to S^2$ (see figure) has nonzero degree. Birkhoff's argument (or the minimax argument) allows us to conclude that M has a nontrivial closed geodesic of length less than or equal to the length of the longest curve in the 1-parameter family. We will use this



argument in the proofs of the main Theorems 4.1 and 4.2. We will use the higher dimensional version of this in §6 in discussing convex hypersurfaces. Further, we will even use a modification of this argument in the case of noncompact surfaces (§5).

The second idea that we will use is the Birkhoff curve shortening process, B.C.S.P. (which Birkhoff used in the above mentioned argument). Since we need to derive some new properties of B.C.S.P., we will recall it here.

The B.C.S.P., β^N: $\Lambda^E \to \Lambda^E$, depends on an integer parameter N, and is a map from Λ^E, the space of curves of energy less than E, to itself. β^N is called the B.C.S.P. with N breaks. For fixed E, N is chosen so large such that $\sqrt{E}/N$ is smaller than the injectivity radius of the manifold, inj(M), or in some cases the injectivity radius of a part of the manifold. In each application we will choose the number of breaks large enough to suit our purposes.

Given a curve $\gamma \in \Lambda^E$ we will define the new curve $\beta^N(\gamma) \in \Lambda^E$ as well as a homotopy γ_s, $s \in [0,1]$, from $\gamma = \gamma_0$ to $\beta^N(\gamma) = \gamma_1$. The homotopy will be defined in such a way that $L(\gamma_{S_2}) \leqslant L(\gamma_{S_1})$ whenever $S_2 \geqslant S_1$.

We will assume that γ is parametrized proportional to arclength. If not the first part of the homotopy is to reparametrize γ so that it is. We define $\gamma_{1/2}$ to be the unique piecewise geodesic closed curve such that $\gamma_{1/2}(i/N) = \gamma(i/N)$ for all integers $i = 0, 1, 2, \cdots, N$, and such that $\gamma_{1/2}|_{[i/N,(i+1)/N]}$ is a minimizing geodesic parametrized proportional to arclength. The uniqueness of $\gamma_{1/2}$ comes from:

$$d\left(\gamma\left(\frac{i}{N}\right), \gamma\left(\frac{i+1}{N}\right)\right) \leqslant L\left(\gamma|_{[i/N,(i+1)/N]}\right) \leqslant \frac{L(\gamma)}{N} \leqslant \frac{\sqrt{E}}{N} < \text{inj}(M).$$

For $s \in [0, \frac{1}{2}]$ we define γ_s by

$$\gamma_s\left(\frac{i}{N} + t\right) = \begin{cases} \tau_i^s(t) & 0 \leqslant t \leqslant \dfrac{2s}{N}, \\ \gamma\left(\dfrac{i}{N} + t\right) & \dfrac{2s}{N} \leqslant t \leqslant \dfrac{1}{N}, \end{cases}$$

where τ_i^s is the minimzing geodesic from $\gamma(i/N)$ to $\gamma(i/N + 2s/N)$ parametrized on the interval $[0, 2s/N]$ proportional to arclength. The uniqueness and continuity of the γ_s follows (as before) from the fact that $L(\tau_i^s) < \text{inj}(M)$ for all i and s.

γ_1 is defined as the unique piecewise geodesic closed curve with $\gamma_1(i/N + 1/(2N)) = \gamma_{1/2}(i/N + 1/(2N))$ which is parametrized proportional to arclength on each interval $[i/N + 1/(2N), (i + 1)/N + 1/(2N)]$. We then define γ_s for $s \in [\frac{1}{2}, 1]$ to homotope between $\gamma_{1/2}$ and γ_1 in the same way that γ_s, $s \in [0, \frac{1}{2}]$ homotopes from γ_0 to $\gamma_{1/2}$. The continuity and uniqueness follow as before.

Birkhoff shows that β^N: $\Lambda^E \to \Lambda^E$ is continuous if N is large in terms of E and the geometry of M (we will not need this fact directly). The closed geodesics are the only fixed points of β^N. Birkhoff also shows that for any $\gamma \in \Lambda^E$ the sequence $\{\gamma_i\}$, defined by $\gamma = \gamma_0$ and $\gamma_{i+1} = \beta^N(\gamma_i)$, converges to a closed geodesic. However, this closed geodesic may be a point curve. If the limit is a point curve then the homotopies described above give rise in a natural way to a map from the two disk D^2 into M with γ as the boundary. We should remark that since the first step in B.C.S.P. is to reparametrize proportional to arclength and since $\beta^N(\gamma)$ will not in general be globally parametrized by arclength parts of the map from D^2 into M will consist of simply reparametrizing these curves.

For the rest of this section let M be a complete connected oriented surface (two dimensional). Let $\gamma \in \Lambda$ be a simple (no self intersections) closed curve on M which divides M into two components. Let Ω (open) be one of these components. Then γ will be called *convex to* Ω if there is an $\varepsilon > 0$ such that for all $x, y \in \gamma$, with $d(x, y) < \varepsilon$, the minimizing geodesic τ from x to y satisfies $\tau \subset \overline{\Omega}$. In fact this means that if $x, y \in \overline{\Omega}$ with $d(x, y) < \varepsilon$ then $\tau \subset \overline{\Omega}$. For if not there would be points $\bar{x}, \bar{y} \in \gamma \cap \tau$ such that the segment $\bar{\tau}$ of τ from $\bar{x}$ to $\bar{y}$ does not lie in $\overline{\Omega}$. But $d(\bar{x}, \bar{y}) = L(\bar{\tau}) < L(\tau) < \varepsilon$. Hence, by the definition of ε, $\bar{\tau} \subset \overline{\Omega}$ yielding a contradiction.

In the applications in this paper γ will be a piecewise geodesic curve. In this case, the above definition reduces to the condition that all the angles of γ are convex to Ω. In general it means that, in addition to the angles being convex to Ω, at the C^∞ points of γ the curvature vector points toward Ω.

For $x \in M$, we let $\text{inj}(x)$ represent the injectivity radius at x (i.e. the minimum distance from x to its cut locus). For a compact set K we let $\text{inj}(K) = \min\{\text{inj}(x) \mid x \in K\}$.

For γ convex to Ω as above we define $\varepsilon(\gamma) > 0$ as follows. For each $x \in \gamma$ we let $\gamma_x = \gamma|_{[a,b]}$ be the largest connected open segment of γ containing x

such that for all $y \in \gamma_x$, $d(x, y) < \text{inj}(\gamma)$. Let $\varepsilon(x, \gamma) = \frac{1}{2}d(x, \gamma - \gamma_x)$ (if $\gamma = \gamma_x$ then set $\varepsilon(x, \gamma) = \frac{1}{2}\text{inj}(\gamma)$). We then define

$$\varepsilon(\gamma) = \min\{\varepsilon(x, \gamma) \mid x \in \gamma\}.$$

Note that $\varepsilon(\gamma) \leqslant \frac{1}{2}\text{inj}(\gamma)$.

We now introduce an elementary lemma. The proof is straightforward but is included for completeness.

Lemma 2.1. *Let γ be convex to Ω. Then*

1) *For $x \in \gamma$ and $y \in \gamma_x$ the unique minimizing geodesic τ from x to y satisfies $\tau \subset \overline{\Omega}$ and either $\tau \cap \partial\Omega = \{x, y\}$ or $\tau \subset \partial\Omega$.*

2) *for $x \in \gamma$ and $y \in \overline{\Omega}$ such that $d(x, y) < \varepsilon(\gamma)$, the unique minimizing geodesic τ from x to y satisfies $\tau \subset \overline{\Omega}$.*

3) *Let $x \in \gamma$ and $y \in \Omega$. Then if τ is the shortest path in $\overline{\Omega}$ from x to y then τ is a geodesic of M, $\tau \cap \partial\Omega = \{x\}$, and $\tau'(0)$ is not tangent to $\gamma = \partial\Omega$.*

4) *Let $x, y \in \Omega$ and τ the shortest path in $\overline{\Omega}$ from x to y. Then $\tau \subset \Omega$.*

Proof. We start by noting that if τ is a geodesic segment such that $\tau \subset \overline{\Omega}$ and $\tau'(x_0)$ is tangent to $\gamma = \partial\Omega$ for some $x_0 \in \gamma$ then the convexity of γ implies that $\tau \subset \partial\Omega = \gamma$. In fact, if x_0 is not a C^∞ point of γ then the above holds under the assumption that $\tau'(x_0)$ is tangent to either of the tangents of γ.

The second part of (1) follows from the first part (i.e. $\tau \subset \overline{\Omega}$) and the above by noting that if an interior point of τ intersects γ then it must be tangent to γ at that point since the angles of γ are convex to Ω.

To see the first part of (1), we assume for simplicity that $x = \gamma(0)$, $y = \gamma(a)$ and for all $0 \leqslant t \leqslant a$, $d(x, \gamma(t)) < \text{inj}(\gamma)$. Let τ_t be the unique minimizing geodesic from x to $\gamma(t)$. Since $d(x, \gamma(t)) < \text{inj}(\gamma)$, τ_t varies continuously with t. Let $\bar{t} = \sup\{t \in [0, a] \mid \tau_s \subset \overline{\Omega} \text{ for } s \leqslant t\}$. By the definition of convexity $\bar{t} > 0$. By the continuity of τ_t we have $\tau_{\bar{t}} \subset \overline{\Omega}$. Assume $\bar{t} \neq a$. There are two cases. In the first case $\tau_{\bar{t}} \subset \partial\Omega$. In this case the convexity of $\partial\Omega$ makes it clear that $\tau_{\bar{t}+\delta} \subset \overline{\Omega}$ for small δ contradicting the definition of $\bar{t}$. In the other case $\tau_{\bar{t}} \cap \partial\Omega = \{x, \gamma(\bar{t})\}$. Assume τ_t: $[0, 1] \to M$ is parametrized proportional to arclength and has length $l(t)$. Let $\varepsilon > 0$ be less than the ε in the definition of the convexity of γ and less than $l(\bar{t})/3$. Since $\tau_{\bar{t}}[\varepsilon/l(\bar{t}), 1 - \varepsilon/l(\bar{t})] \subset \Omega$ we can choose $\delta > 0$ so that for all $\bar{t} < t < \bar{t} + \delta$, $\tau_t[\varepsilon/l(t), 1 - \varepsilon/l(t)] \subset \Omega$. Further for such t $\tau_t[0, \varepsilon/l(t)] \subset \Omega$ since $\tau_t(0), \tau_t(\varepsilon/l(t)) \in \overline{\Omega}$, $d(\tau(0), \tau_t(\varepsilon/l(t))) = \varepsilon$, and the definition of ε. Similarly $\tau_t[1 - \varepsilon/l(t), 1] \subset \overline{\Omega}$. Thus we have $\tau_t \subset \overline{\Omega}$ contradicting the maximality of $\bar{t}$. Thus we have shown 2.1.1.

To see 2.1.2, let τ be the unique (since $\varepsilon(\gamma) \leqslant \frac{1}{2}\operatorname{inj}(\gamma)$) geodesic from x to y. If $\tau \not\subset \bar{\Omega}$ then there is an interval $[t_0, t_1]$ such that $\tau(t_0, t_1) \subset M - \bar{\Omega}$ while $\tau(t_0),\ \tau(t_1) \in \bar{\Omega}$. But $L(\tau) < \varepsilon(\gamma) \leqslant \varepsilon(\tau(t_0), \gamma)$ means $\tau(t_1) \in \gamma_{\tau(t_0)}$ which contradicts 2.1.1.

For 2.1.3 and 2.1.4 we see that if τ is a minimizing curve in $\bar{\Omega}$ then $\tau \cap \Omega$ is a geodesic of M. Further $\overline{\tau \cap \Omega}$ can only intersect $\partial\Omega$ at an endpoint of τ. To see this we note that τ cannot be tangent to $\partial\Omega$ unless $\tau \cap \Omega = \varnothing$ (i.e. $\tau \subset \partial\Omega$) as mentioned before, but on the other hand if an angle is made τ can be shortened by "cutting the corner". The convexity of γ allows this "cutting the corner" to happen through curves in $\bar{\Omega}$ even at non-C^∞ points. 2.1.3 and 2.1.4 now follow easily.

Lemma 2.2. *Let γ be convex to Ω and have length L. Assume $\bar{\Omega}$ is compact and let $N > L/\operatorname{inj}(\bar{\Omega})$ (also $N \gg 2$). Then if we apply B.C.S.P. with N breaks to γ the resulting curves γ_t satisfy*:

(1) $\gamma_t \subset \bar{\Omega}$,

(2) γ_t *is simple and convex to* $\Omega_t \equiv \Omega - \{x \in \gamma_s \mid 0 \leqslant s \leqslant t\}$.

Proof. We can assume γ is parametrized proportional to arclength for if not we can homotope the parameter to make it so. Each γ_t, $t \in [0, \frac{1}{2}]$ consists of segments of γ and minimizing geodesic segments τ_i^t between $\gamma(i/N)$ and $\gamma(i/N + 2t/N)$. Since $L(\gamma|_{[i/N, i/N+2t/N]}) \leqslant L/N < \operatorname{inj}(\bar{\Omega}) \leqslant \operatorname{inj}(\gamma)$, Lemma 2.1.1 says $\tau_i^t \subset \bar{\Omega}$. Further by 2.1.1 $\tau_i^t \cap \gamma = \{\gamma(i/N), \gamma(i/N + 2t/N)\}$ or $\tau_i^t = \gamma|_{[i/N, i/N+2t/N]}$ (we note that τ cannot coincide with the other arc of γ because it is too long since $N > 2$). To see that γ_t is simple we need only see that τ_i^t intersects τ_j^t only at common endpoints if $i \neq j$. Since they are both minimizing geodesics they can intersect at most once. But consider the open set $\Omega_i^t \subset \Omega$ bounded by $\tau_i^t \cup (-\gamma|_{[i/N, i/N+2t/N]})$ (this is empty if $\tau_i^t \subset \gamma = \partial\Omega$). If τ_j^t intersects τ_i^t at interior points of τ_i^t and τ_j^t then it must intersect transversely and hence enter Ω_i^t (in this case Ω_i^t of course cannot be empty). τ_j^t must thus leave Ω_i^t again but since it cannot intersect τ_i^t again and does not intersect $\gamma|_{[i/N, i/N+2t/N]}$ we get a contradiction. Thus γ_t is simple. Since γ is convex to Ω and $\tau_i^t \subset \Omega$ we see that the angles of γ_t are convex to $\Omega - \bigcup_{i=1}^N \overline{\Omega_i^t}$. Now $\overline{\Omega_i^t} = \{x \in \tau_i^s \mid 0 \leqslant s \leqslant t\}$ by the convexity of $\overline{\Omega_i^t}$ and the fact that $\overline{\Omega_i^t}$ lies inside the injectivity radius of $\gamma(i/N)$. (The fact that $\tau_i^s \subset \overline{\Omega_i^t}$ follows from the proof of Lemma 2.1.1.) Thus we see that γ_t is convex to Ω_t.

The proof for $t \in [\frac{1}{2}, 1]$ follows in exactly the same way since $\operatorname{inj}(\bar{\Omega}) \leqslant \operatorname{inj}(\gamma_{1/2})$.

Remark. If K is a compact set and N is chosen such that $N > L/\operatorname{inj}(K)$ then the above lemma holds as long as $\gamma_t \subset K$ even if Ω is not assumed to be compact.

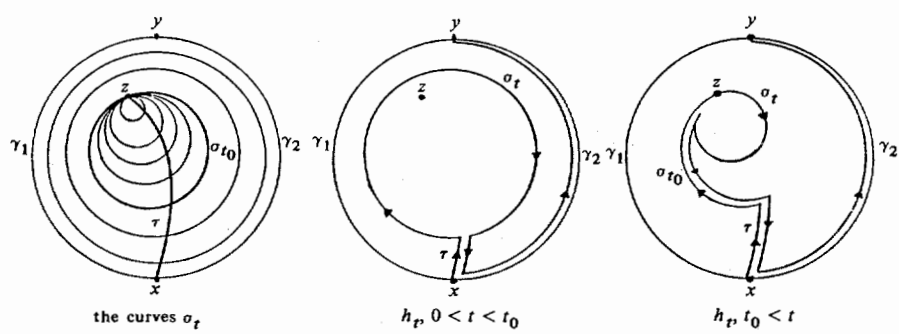
3. The Main Lemmas

We now prove the lemma which is the heart of this paper.

Lemma 3.1. *Let γ_1 and γ_2 be two piecewise smooth curves from x to y such that $\gamma_1 \cup -\gamma_2$ forms a simple closed curve which is convex to an open disk Ω. Assume further that for every $z \in \Omega$, $d_{\bar{\Omega}}(x, z) \leqslant D$, where $d_{\bar{\Omega}}$ represents the distance as measured in $\bar{\Omega}$ and D is some real number. Then either there is a nontrivial closed geodesic lying in $\bar{\Omega}$ of length less than or equal to $L = L(\gamma_1 \cup -\gamma_2)$ or γ_1 is homotopic to γ_2 through curves from x to y lying in $\bar{\Omega}$ of length $\leqslant 3L + 2D$.*

Proof. Assume there is no nontrivial closed geodesic in $\bar{\Omega}$ of length less than or equal to L. Applying B.C.S.P. repeatedly to $\gamma_1 \cup -\gamma_2$ and using Lemma 2.2 we get a homotopy σ_t, $t \in [0,1]$ from $\sigma_0 = \gamma_1 \cup -\gamma_2$ to a point curve σ_1 (say $\sigma_1 = \{z\}$). Further each σ_t is convex to Ω_t ($\Omega_0 = \Omega$) with $\Omega_t \subset \Omega_s$ for $t > s$. In particular $z \in \bar{\Omega}_t$ for all $t \in [0,1)$. Let $t_0 = \min\{t \mid z \in \partial\Omega_t = \sigma_t\}$. It is clear that for $t < t_0$, $z \in \Omega_t$ and for $t \geqslant t_0$, $z \in \sigma_t$. Let τ: $[0,1] \to M$ be a minimizing path from x to z in $\bar{\Omega}$. τ will be a geodesic of M if $t_0 \neq 0$ by Lemma 2.1.3, and we will have no need of τ if $t_0 = 0$.

We claim that for all $t < t_0$, $\tau \cap \sigma_t$ is a single point z_t. The fact that $x \notin \Omega_t$ and $z \in \Omega_t$ implies that $\tau \cap \sigma_t$ is not empty. By Lemma 2.1.3 we need only show that $\tau(s) \in \sigma_t$ implies $\tau|_{[s,1]} \subset \bar{\Omega}_t$. We now fix $s \in [0,1]$. We let $t_s = \min\{t \mid \tau(s) \in \sigma_t\}$, $t^s = \max\{t \mid \tau(s) \in \sigma_t\}$, and $\bar{t} = \sup\{t \mid \tau|_{[s,1]} \subset \bar{\Omega}_t\}$. To prove the claim we need to show $\bar{t} \geqslant \min\{t^s, t_0\}$. We can assume $\bar{t} < t_0$. We see by continuity that $\tau|_{[s,1]} \subset \bar{\Omega}_{\bar{t}}$ and by the maximality of $\bar{t}$ that $\tau|_{[s,1]} \cap \sigma_{\bar{t}} \neq \varnothing$. By Lemma 2.1.4 $\tau(s) \in \bar{\Omega}_{\bar{t}}$ and hence we see $t^s \geqslant \bar{t} \geqslant t_s$. Choose $0 < \varepsilon < \varepsilon(\sigma_{\bar{t}})$ and s_0 such that $s < s_0 < 1$ and $L(\tau|_{[s,s_0]}) < \varepsilon$. Since $\tau|_{[s_0,1]} \subset \Omega_{\bar{t}}$ there is a $\delta > 0$ such that $\tau|_{[s_0,1]} \subset \Omega_t$ and $\varepsilon(\sigma_t) > \varepsilon$ for all t with $\bar{t} \leqslant t < \bar{t} + \delta$. Thus for all t, $\bar{t} \leqslant t \leqslant \min\{t^s, t_0, \bar{t} + \delta\}$ we have $\tau|_{[s,s_0]} \subset \bar{\Omega}_t$ by Lemma 2.1.2, and hence $\tau|_{[s,1]} \subset \bar{\Omega}_t$. Thus the maximality of $\bar{t}$ forces $\bar{t} = \min\{t^s, t_0\}$ and the claim follows.



We are now ready to define the homotopy. First homotope γ_1 to $\gamma_1 \cup -\gamma_2 \cup \gamma_2$ through curves of length $< 2L$. Now for each $t \in [0, t_0)$ let $s(t)$ be the unique value of s such that $\tau(s(t)) \in \sigma_t$. Let $s(t_0) = \sup\{s(t) \mid t \in [0, t_0)\}$. We homotope $\gamma_1 \cup -\gamma_2 \cup \gamma_2$ to $\tau|_{[0,s(t_0)]} \cup \sigma_{t_0} \cup -\tau|_{[0,s(t_0)]} \cup \gamma_2$ through the curves $h_t = \tau|_{[0,s(t)]} \cup \sigma_t \cup -\tau|_{[0,s(t)]} \cup \gamma_2$, where σ_t represents going once around σ_t starting and ending at $\tau(s(t))$. The length of h_t is less than $D + L + D + L = 2L + 2D$. Let $\tau_{t_0} = \tau|_{[0,s(t_0)]}$ and $\bar{\sigma}_{t_0}$ represents the shortest arc of σ_0 from $\tau(s(t_0))$ to z. We now homotope $\tau_{t_0} \cup \sigma_{t_0} \cup -\tau_{t_0} \cup \gamma_2$ to $\tau_{t_0} \cup \bar{\sigma}_{t_0} \cup \sigma_{t_0} \cup -\bar{\sigma}_{t_0} \cup -\tau_{t_0} \cup \gamma_2$, where here σ_{t_0} represents going once around σ_{t_0} starting and ending at z, through curves of length $< D + L/2 + L + L/2 + D + L = 3L + 2D$. This curve is in turn homotopic to $\tau_{t_0} \cup \bar{\sigma}_{t_0} \cup -\bar{\sigma}_{t_0} \cup -\tau_{t_0} \cup \gamma_2$ via the curves $h_t = \tau_{t_0} \cup \bar{\sigma}_{t_0} \cup \sigma_t \cup -\bar{\sigma}_{t_0} \cup -\tau_{t_0} \cup \gamma_2$ for $t \in [t_0, 1]$, whose lengths are less than $3L + 2D$. This last curve is clearly homotopic to γ_2 through curves of length less than $2L + 2D$ and the lemma follows.

Remark 1. $3L + 2D$ is not optimal. One could probably improve this to $\frac{3}{2}L + 2D$ with a little work, but examples show one cannot expect to do much better. As neither estimate leads to sharp answers in the theorems we will not worry about it.

Remark 2. One should note (we will use this fact later) that the homotopy defined above defines a map from the disk D^2 to $\bar{\Omega}$ of local degree ± 1 since the generic point will have a single preimage.

We now come to the lemma in which the area A of the manifold enters. It enters in the (coarea) formula

$$A \geqslant \int_a^b L(S(x,t))\, dt$$

where $\infty \geqslant b \geqslant a \geqslant 0$, x is a fixed point in M, and $S(x,t) = \{y \in M \mid d(x,y) = t\}$, i.e. $S(x,t)$ is the "circle" of radius t centered at x. In general $S(x,t)$ need not be very nice, but for generic t (i.e. for all but a closed set of measure 0) $S(x,t)$ is a piecewise smooth disjoint union of Jordan curves. This was shown by Hartman [14, Proposition 6.1] who generalized results of Fiala [11] to the differentiable category (Fiala considered analytic metrics). The above coarea formula can be found, for example, in equations 6.30 and 6.31 of [14].

Lemma 3.2. *Let M be a complete oriented surface of finite area A. Let $x, y, z \in M$ with τ_y^x, τ_z^y, and τ_x^z minimizing geodesics connecting the respective points. Then*:

(1) *If $w \in \tau_y^x$ is such that $d(w,x) > \sqrt{A}$ and $d(w,y) > \sqrt{A}$, then there is a closed curve through w which is essential in $M - \{x, y\}$ and has length $\leqslant 2\sqrt{A}$.*

(2) *Let* $d_x = d(x, \tau_z^y)$ *(similarly for* d_y *and* d_z*). If* $d_x > \sqrt{2A}$, $d_y > \sqrt{2A}$ *and* $d_z > \sqrt{2A}$, *then there is a nontrivial closed geodesic of length* $\leqslant \sqrt{8A}$.

(3) *If* M *is compact,* $d(x, y) = D(M)$, *the diameter of* M, *and* $d_z > 2\sqrt{2A}$, *then there is a nontrivial closed geodesic of length* $\leqslant \sqrt{8A}$.

(4) *In the case where* M *is diffeomorphic to* $S^1 \times \mathbf{R}^1$ *and* γ *is a line in* M (γ *minimizes distance between any two of its points*) *and for some* z, $d(z, \gamma) > \sqrt{2A}$ *then there is a nontrivial closed geodesic of length* $\leqslant \sqrt{8A}$.

Proof. We begin by noting that for a piecewise smooth simple closed curve σ on a complete surface M, in particular for a component of $S(x, t)$ for fixed x and generic t, either σ is essential in M or σ splits M into two pieces M_1 and M_2. In the latter case σ is essential in $M - \{x_1, x_2\}$ for $x_1 \in M_1$ and $x_2 \in M_2$. One sees this by assuming that σ does not split M. In which case one can create a closed curve τ intersecting σ transversely exactly once. Hence the intersection number modulo 2 of σ with τ is nonzero and σ is essential. In the case where σ divides M, choose τ to be a curve from x_1 to x_2 intersecting σ once transversely. The same argument gives σ essential in $M - \{x_1, x_2\}$. It is easy to see that if M_i is noncompact x_i can be taken to be ∞ (i.e., not included in the removed set).

Let $\tau\colon [0, 1] \to M$ be τ_y^x parametrized by arclength with $\tau(0) = x$, $\tau(L) = y$, and $\tau(t_0) = w$. By assumption $t_0 > \sqrt{A}$ and $L - t_0 > \sqrt{A}$. Since

$$A \geqslant \int_{t_0-\sqrt{A}}^{t_0+\sqrt{A}} L(S(x,t))\,dt \quad \text{and} \quad \int_{t_0-\sqrt{A}}^{t_0+\sqrt{A}} 2\sqrt{A} - 2|t - t_0|\,dt = 2A$$

there is a generic $t \in [t_0 - \sqrt{A}, t_0 + \sqrt{A}]$ such that $L(S(x, t)) \leqslant 2\sqrt{A} - 2|t - t_0|$. Let σ be the component of $S(x, t)$ through $\tau(t)$. σ is a simple closed curve (by the genericity of t) with $L(\sigma) \leqslant 2\sqrt{A} - 2|t - t_0|$ and σ is essential in $M - \{x, y\}$. That σ is essential in $M - \{x, y\}$ follows since either σ is essential in M (hence in $M - \{x, y\}$) or it splits M into two pieces with x in one and y in the other since τ intersects σ transversely exactly once. Thus the curve $\tau|_{[t_0,t]} \cup \sigma \cup -\tau|_{[t_0,t]}$ is a closed curve through w of length $\leqslant 2\sqrt{A}$ and essential in $M - \{x, y\}$.

The first step in the proof of (2) is to note that $d_x + d_y > d(x, y) = L(\tau_y^x) \equiv L$. This is proved by adding the four natural triangle inequalities involving d_x and d_y (for example if $q \in \tau_y^z$ is the closest point to x on τ_z^y, i.e. $d(x, q) = d_x$, then two of the triangle inequalities are $d_x + d(q, z) \geqslant d(x, z)$ and $d_x + d(q, y) \geqslant d(x, y)$). You get strict inequality since the four inequalities cannot be simultaneously equalities (since $z \notin \tau_y^x$). On the other hand $L = d(x, y) \geqslant \max\{d_x, d_y\} > \sqrt{2A}$.

Let $\tau\colon [0, L] \to M$ be τ_y^x with the arclength parameter t. Choose $t_0 \in [0, L]$ such that $\sqrt{A/2} < t_0 < d_x$ and $\sqrt{A/2} < L - t_0 < d_y$, which can be done

since $d_x + d_y > L$ and $d_x, d_y \geqslant \sqrt{2A} > \sqrt{A/2}$. Now $B(x, t_0) \cap B(y, L - t_0)$ has measure 0 so

$$A \geqslant \operatorname{Area}(B(x, t_0)) + \operatorname{Area}(B(y, L - t_0))$$
$$\geqslant \int_0^{\sqrt{A/2}} L(S(x, t_0 - s))\, ds + \int_0^{\sqrt{A/2}} L(S(y, L - t_0 - s))\, ds.$$

Hence as before there is a generic $s \in [0, \sqrt{A/2}]$ such that

$$L(S(x, t_0 - s)) + L(S(y, L - t_0 - s)) \leqslant 2\sqrt{2}\sqrt{A} - 4s$$

and both $S(x, t_0 - s)$ and $S(y, L - t_0 - s)$ are disjoint unions of simple closed curves. Let σ_1 be the component of $S(x, t_0 - s)$ through $\tau(t_0 - s)$ and σ_2 the component of $S(y, L - t_0 - s)$ through $\tau(t_0 + s)$. Since $t_0 < d_x$, $\sigma_1 \cap \tau_y^z = \varnothing$. If σ_1 does not separate M then applying B.C.S.P. repeatedly yields the desired nontrivial (in fact essential) closed geodesic of length $\leqslant \sqrt{8A}$. So we can assume σ_1 separates M which must have x on one side and y and z on the other. σ_1 intersects both τ_z^x and τ_y^x transversely (in fact perpendicularly) in one point. Similarly we can assume σ_2 intersects τ_z^y and τ_y^x transversely once. Define σ to be $\sigma_1 \cup \tau|_{[t_0 - s, t_0 + s]} \cup \sigma_2 \cup -\tau|_{[t_0 - s, t_0 + s]}$. We see that $L(\sigma) \leqslant \sqrt{8A}$. We make sure to choose the orientation of σ_1 and σ_2 so that σ has the form of a figure 8 around x and y, that is so that the oriented intersection number of σ with τ (say in $M - \{x, y\}$) is $+2$ rather than 0. Applying B.C.S.P. repeatedly to σ leads either to a closed geodesic of length $\leqslant L(\sigma) \leqslant \sqrt{8A}$ or to a point curve. But it cannot lead to a point curve for if it did some curve σ_{s_0} in the homotopy would have to pass through a vertex (x, y or z) while still intersecting the opposite geodesic (τ_z^y, τ_z^x, or τ_y^x) which we see by intersection number arguments. Now the fact that $L(\sigma_{s_0}) \leqslant \sqrt{8A}$ and d_x, d_y and $d_z > \frac{1}{2}\sqrt{8A}$ yields the desired contradiction.

Part (3) follows directly from part (2) and triangle inequalities. Let $w \in \tau_z^y$ be such that $d(x, w) = d_x$. Then $d_x + d(w, y) \geqslant d(x, y) = D \geqslant d(z, y)$ and $d_x + d(w, z) \geqslant d(x, z) \geqslant 2\sqrt{2A}$. Adding the above gives $d_x > \sqrt{2A}$. Similarly $d_y > \sqrt{2A}$.

Part (4) also follows from part (2). Choose $w \in \gamma$ such that $d(z, w) = d(z, \gamma)$ and choose x and y on γ such that $d(w, y) > d(z, w) + \sqrt{2A}$, $d(w, x) > d(z, w) + \sqrt{2A}$ and w is between x and y, i.e. $d(x, y) = d(x, w) + d(w, y)$. We therefore have $d(x, w) + d(w, y) = d(x, y) \leqslant d_x + d(z, y) \leqslant d_x + d(z, w) + d(w, y)$. Hence $d_x \geqslant d(x, w) - d(z, w) > \sqrt{2A}$. Similarly $d_y > \sqrt{2A}$. Since we have by assumption $d_z > \sqrt{2A}$ part (2) yields part (4).

We now study further the case of Lemma 3.2.1. We will consider the case of a geodesic segment τ in M; we are interested in three cases in particular. The first is when M is diffeomorphic to S^2 and τ is a minimizing geodesic. The

second is when M is diffeomorphic to $\mathbf{R}^2$ and τ is a ray (i.e. τ: $[0, \infty) \to M$ and τ is a minimizing geodesic between any two points on it). The third is when M is a cylinder $S^1 \times \mathbf{R}^1$ and τ is a line (i.e. τ: $(+\infty, \infty) \to M$ and it minimizes between any two points on it). We will speak of τ as a minimizing geodesic from x to y but either one or both of x and y will represent ∞ in the second and third cases above.

Lemma 3.3. *Let M be one of the three cases above and τ a minimizing geodesic from x to y (as discussed above). Let $w \in \tau$ be such that $d(w, y) > \sqrt{A}$ and $d(w, x) > \sqrt{A}$. Then there is a shortest closed curve γ through w which is essential in $M - \{x, y\}$. If γ_1 and γ_2 are two such shortest curves we have*:

(1) γ_i: $[0, l_w] \to M$ *is a simple closed geodesic loop (not necessarily smooth at w) at w of length $l_w \leqslant 2\sqrt{A}$.*

(2) $\gamma_i \cap \tau = \{w\}$ *and the vectors $\gamma_i'(0)$ and $-\gamma_i'(l_w)$ lie on opposite sides of τ.*

(3) $\gamma_i \cap \gamma_j = \{w\}$ *or $\gamma_i \equiv \gamma_j$.*

(4) *Assume further that both γ_i separate M into two pieces. Let Ω_i be the component of $M - \gamma_i$ such that γ_i is convex to Ω_i (this must be true for at least one component for γ_i has but one angle). Then either $\Omega_i \cap \Omega_j = \varnothing$, $\Omega_i \subset \Omega_j$, or $\Omega_j \subset \Omega_i$.*

Proof. By Lemma 3.2.1 there are such short curves of length $\leqslant 2\sqrt{A}$. Since w is further from x or y than $\sqrt{A}$ a shortest such curve cannot pass through x or y hence must be a geodesic loop γ_w. The above holds in all cases even though x or y may be ∞, by choosing points close to ∞ (after fixing w) and then applying Lemma 3.2.1. We now see that $\gamma_w \cap \tau = \{w\}$, for if not we could replace one arc of γ_w with a segment of τ reducing the length (since τ minimizes from x to y) and staying essential in $M - \{x, y\}$ (for the correct choice of arc to replace). Now if both vectors $\gamma_w'(0)$ and $-\gamma_w'(l_w)$ lie on the same side of τ then the fact that $\tau \cap \gamma_w = \{w\}$ implies that γ_w can be homotoped to miss τ. Since $M - \tau$ is simply connected and $M - \{x, y\} \supset M - \tau$ we see γ_w cannot be essential in $M - \{x, y\}$. Hence $\gamma_w'(0)$ and $-\gamma_w'(l_w)$ lie on opposite sides of τ as claimed. Now assume γ_w was not simple. By throwing away part of γ_w one can construct a closed curve $\bar{\gamma}_w$ through w which is shorter than γ_w but whose intersection with τ is the same as γ_w's, that is $\bar{\gamma}_w$ intersects τ transversely in one point. Thus $\bar{\gamma}_w$ is essential in $M - \{x, y\}$. This contradicts the minimality of γ_w and hence γ_w is simple. This proves (1) and (2).

Assume that γ_w and $\bar{\gamma}_w$ are two such loops. Orient them so that $\gamma_w'(0)$ and $\bar{\gamma}_w'(0)$ lie on the same side of τ. If γ_w and $\bar{\gamma}_w$ intersect then one can construct two closed curves τ_1 and τ_2 through w as follows: τ_1 starts at w follows γ_w to the point of intersection and then follows $\bar{\gamma}_w$ back to w. τ_2 does the opposite.

Both τ_1 and τ_2 intersect τ only at w and are transverse there and hence are essential in $M - \{x, y\}$. Now $L(\tau_1) + L(\tau_2) = L(\gamma_w) + L(\bar{\gamma}_w) = 2l_w$ so one of τ_1 or τ_2, say τ_i, satisfies $L(\tau_i) \leqslant l_w$. But τ_i is not smooth at the intersection point of γ_w with $\bar{\gamma}_w$ and hence can be shortened contradicting the definition of l_w. Thus (3) follows.

From (3) we see that $\gamma_i - \{w\} \subset \Omega_j$ or $\gamma_i - \{w\} \subset M - \bar{\Omega}_j$. We can tell which of these two cases happens by looking near w. Look at the angle between $\gamma_j'(0)$ and $-\gamma_j'(l_w)$ (the one that is less than π or in the case of the angle equal to π the one containing Ω_j). If both $\gamma_i'(0)$ and $-\gamma_i'(l_w)$ lie in this angle then $\gamma_i - \{w\} \subset \Omega_j$. If not, they must both be outside since $\gamma_i - \{w\} \subset M - \bar{\Omega}_j$. It is not hard to see that if $\gamma_i - \{w\} \subset M - \bar{\Omega}_j$ and $\gamma_j - \{w\} \subset M - \bar{\Omega}_i$ then $\Omega_i \cap \Omega_j = \varnothing$. Further if $\gamma_i - \{w\} \subset \Omega_j$ and $\gamma_j - \{w\} \subset M - \bar{\Omega}_i$ then $\Omega_i \subset \Omega_j$. Thus we need only consider the case $\gamma_j - \{w\} \subset \Omega_i$ and $\gamma_i - \{w\} \subset \Omega_j$. But this cannot happen since it says $\gamma_i'(0)$ and $-\gamma_i'(l_w)$ lie between $\gamma_j'(0)$ and $-\gamma_j'(l_w)$ and vice versa.

This concludes the proof of the lemma.

4. The Main Theorems

In this section we consider the case where M is diffeomorphic to S^2.

Theorem 4.1. *Let M be a riemannian manifold, diffeomorphic to S^2, of diameter D. Then $L \leqslant 9D$, where L is the length of the shortest nontrivial closed geodesic on M.*

Proof. Choose $x, y \in M$ such that $d(x, y) = D$. Let $\mathfrak{A} = \{\tau \mid \tau$ is a minimizing geodesic from x to $y\}$. Berger's lemma (see [8, p. 106]) says that for every $V \in T_xM$ there is a $\tau \in \mathfrak{A}$ such that $\langle V, \tau'(0)\rangle \geqslant 0$. Similarly for every $W \in T_yM$ there is a $\tau \in \mathfrak{A}$ such that $\langle W, -\tau'(D)\rangle \geqslant 0$. Thus we can pick a finite number of distinct geodesics $\tau_1, \tau_2, \cdots, \tau_n \in \mathfrak{A}$ (in fact it is not hard to see that n can be taken to satisfy $2 \leqslant n \leqslant 4$) such that $\{\tau_1, \tau_2, \cdots, \tau_n\}$ satisfies the same property as $\mathfrak{A}$. We can assume $n \neq 2$ for if so $\tau_1 \cup -\tau_2$ is a closed geodesic of length $2D$. We order them so that the $\tau_i'(0)$ come in order counterclockwise from $\tau_1'(0)$. Since τ_i minimizes length $\tau_i \cap \tau_j = \{x, y\}$ for $i \neq j$. Since for every $V \in T_xM$ there is an i such that $\langle V, \tau_i'(0)\rangle \geqslant 0$ we see that $\sphericalangle(\tau_i'(0), \tau_{i+1}'(0) \leqslant \pi$, where the angle is measured in the counterclockwise sense (here $n + 1$ is the same as 1). Similarly $\sphericalangle(-\tau_{i+1}'(D), -\tau_i'(D)) \leqslant \pi$. In particular $\tau_i \cup -\tau_{i+1}$ is a simple closed curve which is convex to the domain Ω_i lying between them (in the obvious way). If $z \in \Omega_i$ then the minimizing geodesic from z to y must lie in $\bar{\Omega}_i$ by the minimality of τ_i and τ_{i+1}. Thus by Lemma 3.1 either there is a closed geodesic of length $\leqslant 2D$ or $-\tau_i$ is

homotopic to $-\tau_{i+1}$ through curves of length $\leqslant 3L + 2D = 8D$ lying in Ω_i. We now describe a short homotopy from the point curve $\{x\}$ to the point curve $\{y\}$:

$$\{x\} \sim (\tau_1 \cup -\tau_1) \sim (\tau_1 \cup -\tau_2) \sim \cdots$$
$$\sim (\tau_1 \cup -\tau_n) \sim (\tau_1 \cup -\tau_1) \sim \{y\}$$

where the homotopy from $-\tau_i$ to $-\tau_{i+1}$ is through curves in $\overline{\Omega}_i$ as in Lemma 3.1. Remark 2 following Lemma 3.1 shows that the induced map from S^2 to S^2 has degree 1. Hence by Birkhoff's idea there is a nontrivial closed geodesic of length less than the length of the longest curve in the homotopy, i.e., less than $9D$.

Remark. $9D$ is clearly not the best constant for this theorem. In fact by improving Lemma 3.1 as suggested in the remark following it one could improve this to $6D$ but this is also unlikely to be sharp.

Theorem 4.2. *Let M be a riemannian manifold, diffeomorphic to S^2, of area A. Then $L \leqslant 31\sqrt{A}$, where L is the length of the shortest nontrivial closed geodesic.*

Proof. By Theorem 4.1 we can assume $D > \frac{31}{9}\sqrt{A} > 2\sqrt{A}$. Let $x, y \in M$ such that $d(x, y) = D$. Let τ be a minimizing geodesic from x to y parametrized by arclength τ: $[0, D] \to M$. For each $t \in (\sqrt{A}, D - \sqrt{A})$ there is a simple geodesic loop γ_t (not necessarily unique) through $\tau(t)$ as in Lemma 3.3. γ_t separates M into pieces Ω_x, Ω_y with $x \in \Omega_x$ and $y \in \Omega_y$ and, since it is a geodesic loop, is convex to Ω_x or Ω_y. For each $t \in (\sqrt{A}, D - \sqrt{A})$ we say $t \in S_x$ if there is a γ_t as in Lemma 3.3 with γ_t convex to Ω_x. Similarly define S_y. It is easy to see that both S_x and S_y are closed subsets of $(\sqrt{A}, D - \sqrt{A})$. It follows from the fact that a sequence of geodesic loops has a convergent subsequence to a geodesic loop (the resulting loop can't pass through x or y for length reasons). Thus either $S_x \cap S_y \neq \varnothing$ or one of S_x or S_y is empty.

We first consider the case where $S_x \cap S_y \neq \varnothing$. Let $t_0 \in S_x \cap S_y$. This means there are geodesic loops γ_x and γ_y through $\tau_y^x(t_0)$ with γ_x convex to Ω_x and γ_y convex to Ω_y, where Ω_x and Ω_y are open with $x \in \Omega_x$ and $y \in \Omega_y$. If $\gamma_x = \gamma_y$ then it is a closed geodesic of length $\leqslant 2\sqrt{A}$ (by Lemma 3.3) and the theorem follows. If not then Lemma 3.3 tells us that $\gamma_x \cap \gamma_y = \{\tau_y^x(t_0)\}$ and $\Omega_x \cap \Omega_y = \varnothing$ since $x \notin \Omega_y$ and $y \notin \Omega_x$. Let $\Omega = M - (\overline{\Omega}_x \cup \overline{\Omega}_y)$. Then $\partial\Omega = \gamma_x \cup -\gamma_y$ which is convex to Ω by Lemma 3.3.2. Assuming there are no closed geodesics of length $\leqslant 2\sqrt{A}$ in Ω_x repeated applications of B.C.S.P. and Lemma 2.2 show γ_x is homotopic to a point curve through curves in Ω_x of length $\leqslant 2\sqrt{A}$. Similarly γ_y is homotopic to a point curve through curves in Ω_y of length $\leqslant 2\sqrt{A}$. Further, if there is no closed geodesic in Ω of length

$\leqslant 4\sqrt{A}$, γ_x is homotopic to γ_y through curves in Ω of length $\leqslant 12\sqrt{A} + 2\overline{D}$ by Lemma 3.1, where $\overline{D} = \max\{d_\Omega(z, \tau_y^x(t_0)) \mid z \in \Omega\}$. (As stated one cannot apply Lemma 3.1 directly to $\gamma_x \cup -\gamma_y$ since it is not strictly speaking a simple curve. But after the first application of B.C.S.P. it becomes simple and the argument carries through.)

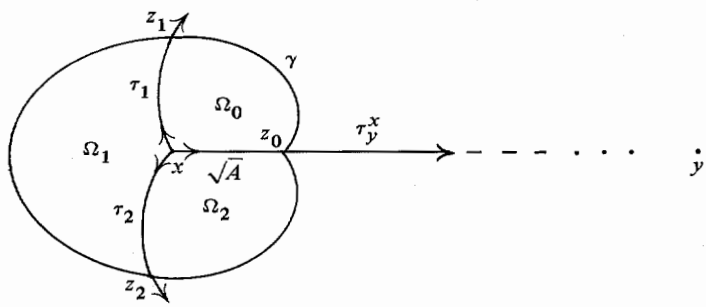
By putting these three homotopies together we get a one parameter family of curves from a point curve to a point curve such that the induced map from S^2 to S^2 has degree 1 (see the remark following 3.1). Thus the Birkhoff method yields a nontrivial closed geodesic of length $\leqslant 12\sqrt{A} + 2\overline{D}$.

We thus need to bound $\overline{D}$. Let $z \in \Omega$. By Lemma 3.2.3 we may assume that $d(z, \tau_y^x) \leqslant 2\sqrt{2A}$. Since $\tau_y^x \cap \gamma_x = \{\tau_y^x(t_0)\}$, $\tau_y^x \cap \gamma_y = \{\tau_y^x(t_0)\}$, $x \in \Omega_x$, and $y \in \Omega_y$ it is easy to see that $\tau_y^x \cap \Omega = \varnothing$. Thus $d(z, \gamma_x \cup -\gamma_y) \leqslant 2\sqrt{2A}$ and $\overline{D} \leqslant (2\sqrt{2} + 1)\sqrt{A}$.

Thus if $S_x \cap S_y \neq \varnothing$ there is a nontrivial closed geodesic of length $\leqslant (14 + 4\sqrt{2})\sqrt{A} < 31\sqrt{A}$.

We now consider the case where one of S_x or S_y is empty. We can assume $S_x = \varnothing$ and $S_y = (\sqrt{A}, D - \sqrt{A})$. In fact there will be a simple closed geodesic loop γ at $\tau(\sqrt{A})$ which is convex to Ω_y, is essential in $M - \{x, y\}$, and has length $\leqslant 2\sqrt{A}$. We find γ as a limit (of a subsequence if necessary) of minimal loops at $\tau(t_i)$ for $t_i \searrow \sqrt{A}$. All of the properties follow immediately once we see that $\gamma \subset M - \{x, y\}$. $y \notin \gamma$ by length considerations. If $x \in \gamma$ then γ must have length $= 2\sqrt{A}$ and both arcs of γ are minimizing geodesics from $\tau(\sqrt{A})$ to x but this cannot be as $\tau|_{[0,\sqrt{A}]}$ is the unique minimizing geodesic from $\tau(0) = x$ to $\tau(\sqrt{A})$ (since it minimizes past $\sqrt{A}$).

Since x is at maximum distance from y we can use Berger's lemma (see [8, p. 106]) to find minimzing geodesics τ_1 and τ_2 from x to y such that $\tau_y^{x\prime}(0)$, $\tau_1'(0)$, and $\tau_2'(0)$ do not lie in an open half plane. (It may happen that there is only one other geodesic τ_1 in which case $\tau_1'(0) = -\tau_y^{x\prime}(0)$. The arguments that follow work equally well in this case but we will make them in the case where τ_1 and τ_2 both exist.)



The geodesic loop γ intersects each of τ_y^x, τ_1, and τ_2 in one point $z_0 = \tau_y^x\sqrt{A}$, z_1, and z_2 respectively (see figure). This is true since if not we would be able to replace a segment of γ with a segment of τ_i decreasing the length and leaving the new curve essential in $M - \{x, y\}$. This would contradict the minimality of the length of the geodesic loops which converge to γ.

We will use the notation $\overline{xz_i}$ and $\overline{z_iz_j}$ to represent the geodesic segments (in figure) between the corresponding points. Note that $\overline{z_iz_j}$ is the appropriate segment of γ and not necessarily a minimizing geodesic.

We know $L(\overline{xz_0}) = \sqrt{A}$, $L(\overline{z_iz_j}) \leqslant 2\sqrt{A}$ and $L(\overline{xz_i}) \leqslant 2\sqrt{A}$ for $i = 1, 2$ since $L(\gamma) \leqslant 2\sqrt{A}$. The geodesic triangles xz_0z_1, xz_1z_2, and xz_2z_0 are convex to the domain Ω_0, Ω_1, and Ω_2 respectively (see figure). We can assume by Lemma 3.2.3 that for every $z \in M$, $d(z, \tau_y^x) \leqslant 2\sqrt{2A}$. If $z \in \Omega_i$ then $d_{\bar{\Omega}_i}(z, x) \leqslant (2\sqrt{2} + 2)\sqrt{A}$, since the minimizing geodesic from z to τ_y^x must hit $\partial\Omega_i$ and any point on $\partial\Omega_i$ is connectable to x along $\partial\Omega_i$ through curves of total length $\leqslant 2\sqrt{A}$. (The last part of the above can be seen as follows: Starting at $w \in \gamma$ one can trace along the short loop of γ to z_0 (length $\leqslant \sqrt{A}$) then follow $\overline{xz_0}$ back to x (length $= \sqrt{A}$). This curve may leave $\partial\Omega_i$ but the curve that starts like this until it hits a z_i and then runs to x along τ_i must be even shorter. Of course if $w \in \partial\Omega_i - \gamma$ one simply follows a τ to x.)

We now create a homotopy from the point curve $\{x\}$ to γ using Lemma 3.1 repeatedly as follows:

$$\begin{aligned}\{x\} &\sim \left(\overline{xz_0}\right) \cup \left(\overline{z_0x}\right) \sim \left(\overline{xz_0}\right) \cup \left(\overline{z_0z_1}\right) \cup \left(\overline{z_1x}\right) \\ &\sim \left(\overline{xz_0}\right) \cup \left(\overline{z_0z_1}\right) \cup \left(\overline{z_1z_2}\right) \cup \left(\overline{z_2x}\right) \\ &\sim \left(\overline{xz_0}\right) \cup \left(\overline{z_0z_1}\right) \cup \left(\overline{z_1z_2}\right) \cup \left(\overline{z_2z_0}\right) \cup \left(\overline{z_0x}\right) \sim \gamma.\end{aligned}$$

The longest curves in this homotopy have length $\leqslant \sqrt{A} + 2\sqrt{A} + 3(6\sqrt{A}) + 2(2\sqrt{2} + 2)\sqrt{A} = (25 + 4\sqrt{2})\sqrt{A} < 31\sqrt{A}$. Since γ is convex to Ω_y we may assume as usual that γ is homotopic to a point curve through curves in Ω_y of length $\leqslant 2\sqrt{A}$. Combining these homotopies the Birkhoff idea (since once again the induced map from S^2 to S^2 has degree 1) yields a nontrivial closed geodesic of length $\leqslant 31\sqrt{A}$. The theorem follows.

5. Noncompact Surfaces of Finite Area

It is known (see [20] and [2]) that every complete surface of finite area has closed geodesics (in fact infinitely many). In this section, using ideas developed in previous sections, we show:

Theorem 5.1. *There is a constant c such that if M is a complete surface (without boundary) of area A then $c\sqrt{A} \geqslant L$, where L is the length of the shortest closed geodesic.*

Proof. By taking oriented double covers we may assume M is orientable. As discussed in the introduction, previous work had reduced the compact case to the case of S^2. Since Theorem 4.2 takes care of the S^2 case we may assume M is not compact. By Theorem 4.4A of [12] we may assume that M is diffeomorphic to $S^2 - \{\text{points}\}$. We treat this as three cases. Case 1 is when M has at least three ends. In Case 2, M is diffeomorphic to $S^1 \times \mathbf{R}^1$ and in Case 3, M is diffeomorphic to $\mathbf{R}^2$.

Case 1. M has at least three ends.

Choose $x_0 \in M$ and $R_0 > 0$ so large that three of the ends of $M - B(x_0, R_0)$ are pairwise disconnected in $M - B(x_0, R_0)$. Choose x_1, x_2, and x_3 one in each end such that $d(x_0, x_i) = 2R_0 + \sqrt{2A}$. Let $\tau_{x_i}^{x_j}$ be minimizing geodesics from x_i to x_j, $i = 1, 2, 3$. If we show $d(x_1, \tau_{x_3}^{x_2}) > \sqrt{2A}$, $d(x_2, \tau_{x_3}^{x_1}) > \sqrt{2A}$ and $d(x_3, \tau_{x_2}^{x_1}) > \sqrt{2A}$ then Lemma 3.2.2 proves the theorem. By the triangle inequality (in fact a sum of two) $2d(x_1, \tau_{x_3}^{x_2}) + d(x_2, x_3) \geqslant d(x_1, x_2) + d(x_1, x_3)$. We also have $4R_0 + 2\sqrt{2A} \geqslant d(x_i, x_j) \geqslant 2R_0 + 2\sqrt{2A}$. Thus $2d(x_1, \tau_{x_3}^{x_2}) + 4R_0 + 2\sqrt{2A} \geqslant 4R_0 + 4\sqrt{2A}$ and hence $d(x_1, \tau_{x_3}^{x_2}) \geqslant \sqrt{2A}$. The same argument for x_2 and x_3 yields the result.

Case 2. M is diffeomorphic to $S^1 \times \mathbf{R}^1$.

Let $\tau(t)$, $t \in (-\infty, \infty)$, be a line in M. (You can find τ by taking a limit of minimizing geodesics γ_i from x_i to y_i where $x_i \to -\infty$ and $y_i \to +\infty$.) We now apply Lemma 3.3. Thus for each t we get a geodesic loop γ_t (not necessarily unique) at $\tau(t)$, of length $L(t) < 2\sqrt{A}$, satisfying (1), (2) and (3) of Lemma 3.3. γ_t separates since it is simple (Lemma 3.3.1) and essential. Hence Lemma 3.3.4 applies.

If γ_{t_0} is convex to $+\infty$ (and not a closed geodesic) then there is an $\varepsilon > 0$ such that $L(t) < L(t_0)$ for all $t \in (t_0, t_0 + \varepsilon)$. One can see this by looking at the curves $\gamma_{t_0}|_{[\delta, L(t_0)-\delta]} \cup \sigma$, where σ is the minimizing geodesic from $\gamma_{t_0}(L(t_0) - \delta)$ to $\gamma_{t_0}(\delta)$. Each of these curves is shorter than γ_{t_0} and is essential for small δ. The set of points where these curves intersect τ (each curve once) includes some interval $\tau((t_0, t_0 + \varepsilon))$ by the convexity assumption. Similarly if γ_{t_0} is convex to $-\infty$ there is an $\varepsilon > 0$ such that $L(t) < L(t_0)$ for all $t \in (t_0 - \varepsilon, t_0)$.

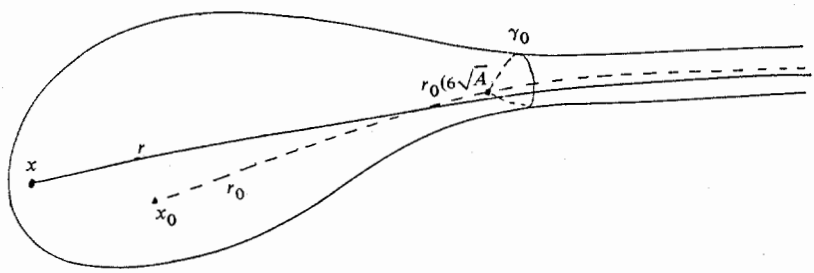
Since the area is finite $L(t)$ cannot be bounded from below as $t \to +\infty$ or as $t \to -\infty$, for if so, then $L(S(x, t))$ would also be bounded below as $t \to \infty$ implying the area is infinite. Thus we see that there is a γ_{t_0} and a γ_{t_1} such that γ_{t_0} is convex to $-\infty$ and γ_{t_1} is convex to $+\infty$. Now by an easy limit argument both $\{t \mid \exists\, \gamma_t \text{ convex to } +\infty\}$ and $\{t \mid \exists\, \gamma_t \text{ convex to } -\infty\}$ are closed. Hence

there is a t_2 in the intersection of these sets giving geodesic loops γ_+ and γ_- at t_2 with γ_- convex to $-\infty$, γ_+ convex to $+\infty$, and satisfying (1), (2), (3) and (4) of Lemma 3.3.

Let Ω be the open disk between γ_+ and γ_-. We may assume by Lemma 3.2.4 that for all $z \in \Omega$, $d(z,\tau) < \sqrt{2A}$. Hence $d_\Omega(z,\tau(t_2)) < \frac{3}{2}\sqrt{2A}$ (follow the minimizing geodesic going from z to τ until it hits $\partial\Omega$ then follow a boundary curve back to $\tau(t_2)$). Thus by Lemma 3.1 we may assume γ_- is homotopic to γ_+ through curves in Ω of length $\leqslant (12 + 3\sqrt{2})\sqrt{A}$.

Now since γ_- and γ_+ are convex to $+\infty$ and $-\infty$ respectively the argument proceeds as usual except since we are in the noncompact case we need to modify B.C.S.P. slightly. This is done in [2, pp. 87–88].

Case 3. M is diffeomorphic to $\mathbf{R}^2$.



Pick $x_0 \in M$ and let $r_0\colon [0,\infty) \to M$ be a ray from x_0. By Lemma 3.3 there is a shortest geodesic loop γ_0 at $r_0(6\sqrt{A})$ of length $\leqslant 2\sqrt{A}$ essential in $M - \{x_0\}$. Let K be the relatively compact (i.e. $\overline{K}$ is compact) component of $M - \gamma_0$. $K \neq \varnothing$ since $x_0 \in K$. Let $x \in K$ maximize the distance to γ_0. So, in particular, $d(x,\gamma_0) \geqslant d(x_0,\gamma_0) \geqslant 5\sqrt{A}$. Let r be a ray from x. Again, by Lemma 3.3, for $t > \sqrt{A}$ there are geodesic loops γ_t through $r(t)$ (not unique as usual) of length $\leqslant 2\sqrt{A}$, essential in $M - \{x\}$, and satisfying Lemma 3.3.

As in the proof of the $S^1 \times \mathbf{R}$ case there are large t with γ_t convex to ∞. The proof, as usual breaks up into two cases. Either all the loops γ_t, $t > \sqrt{A}$, are convex to ∞ or for some $t_0 > \sqrt{A}$ there are two such loops, one convex to x and one convex to ∞.

The case where all γ_t are convex to ∞ is treated as in the S^2 case. Let γ be such a loop at $r(\sqrt{A})$ and let Ω be the relatively compact component of $M - \gamma$. Ω is contained in K since $d(x,\gamma(t)) \leqslant 2\sqrt{A}$ for all t while $d(x,\gamma_0) \geqslant 5\sqrt{A}$. Now for $z \in \Omega$, $d(z,\gamma) + d(\gamma,\gamma_0) \leqslant d(z,\gamma_0) \leqslant d(x,\gamma_0) \leqslant d(x,\gamma) + \sqrt{A} + d(\gamma,\gamma_0) \leqslant 2\sqrt{A} + d(\gamma,\gamma_0)$. Hence for all $z \in \Omega$, $d(z,\gamma) \leqslant 2\sqrt{A}$. Since x is at a local maximum of the distance to γ_0, the proof of Berger's lemma (see [8, p. 106]) yields minimizing geodesics τ_i from x to γ_0 such that for all $V \in T_xM$

there is an i such that $\langle V, \tau_i'(0)\rangle \geqslant 0$. Thus the same proof as in the S^2 case shows that γ is homotopic through short ($\leqslant \text{const}\sqrt{A}$) curves in $\bar{\Omega}$ to the point curve $\{x\}$. The rest of the argument is the modified B.C.S.P. since γ is convex to ∞.

In the other case there are two such geodesic loops, γ_1, and γ_2 at $r(t_0)$ with γ_1 convex to x and γ_2 convex to ∞. Let Ω be the disk between them. The proof would follow as before if we could show that for all $z \in \Omega$, $d(z, \partial\Omega) \leqslant \text{const}\sqrt{A}$. In fact, we will show that if for some $z \in \Omega$, $d(z, \partial\Omega) > 4\sqrt{A}$, then applying B.C.S.P. to $\gamma_1 \cup \gamma_2$ yields a nontrivial closed geodesic of length $\leqslant 4\sqrt{A}$. Thus let $z \in \Omega$ be such that $d(z, \partial\Omega) > 4\sqrt{A}$. We see that $t_0 \geqslant 3\sqrt{A}$, for if $t_0 < 3\sqrt{A}$, $\gamma_2 \subset K$ and $d(z, \gamma_2) \leqslant 4\sqrt{A}$ by the arguments for the previous case. Let σ_1 be a smooth curve from z to ∞ lying in the unbounded component of γ_1 which intersects γ_2 transversely once, and let σ_2 be a smooth curve in the bounded component of γ_2 intersecting γ_1 transversely once from z to x. Now $\gamma_1 \cup \gamma_2$ has intersection number 1 with σ_1 and with σ_2 while it has intersection number 2 with r. Hence if $\gamma_1 \cup \gamma_2$ were to shrink to a point or run off to ∞ under B.C.S.P. then some curve h in the homotopy must either pass through z while still intersecting r or pass through x while still intersecting σ_1. But $d(z, r) \geqslant d(z, \partial\Omega) > 4\sqrt{A}$ and $d(x, \sigma_1) \geqslant d(x, \gamma_1) \geqslant t_0 - \sqrt{A} \geqslant 2\sqrt{A}$. In either case $L(h) \geqslant 4\sqrt{A}$ contradicting $L(h) < L(\gamma_1 \cup \gamma_2) \leqslant 4\sqrt{A}$.

The theorem follows.

6. Convex Hypersurfaces

For $D^n \subset \mathbf{R}^n$ a convex domain and l a line through the origin the width of D in the direction of l is the distance between the two (parallel) tangent spaces to ∂D perpendicular to l. The width of D is the minimum over all directions l. The main theorem in this section leads us to consider the constants:

$$c_0(n) = \inf\{\text{Vol}(D) \mid D^n \subset \mathbf{R}^n \text{ is convex of width } 1\}.$$

Unfortunately the value of $c_0(n)$ is only known in the case $n = 2$. In this case the infimum is achieved for D equal to the equilateral triangle of side $2/\sqrt{3}$ and $c_0(2) = 1/\sqrt{3}$.

It is easy to get a lower bound for $C(n)$, however, the exact value is unknown (see [13, Problem 26]). The best known lower bound for $C_0(n)$ is $2\sqrt{3}/n!$, which is due to Firey [10].

Theorem 6.1. *Let $M^n \subset \mathbf{R}^{n+1}$ be a convex hypersurface and L the length of the shortest nontrivial closed geodesic. Then*

$$2 \cdot \frac{2}{\sqrt[n]{2c_0}} \cdot \sqrt[n]{\text{Vol}(M)} \geqslant L.$$

Remarks. One suspects that the best constant would be $2/\sqrt[n]{2c_0}$ (in which case the theorem is off by a factor of 2) and that equality would hold (in a generalized sense) for M^n = two copies of D^n glued together along ∂D, where D is the best domain in the definition of $c_0(n)$. In particular for $n = 2$ (even without the convexity assumption) one suspects the best constant to be achieved only by the bi-equilateral triangle (two equilateral triangles glued along the boundary). It was pointed out by Calabi that this (degenerate) convex two manifold has a simple and a nonsimple closed geodesic of shortest length. Recently Calabi has shown that the shortest closed geodesic on a convex (nondegenerate) surface is in fact simple. In the nonconvex case it is easy to find examples where the shortest closed geodesic is not simple.

To prove the theorem we need

Lemma 6.2. *Let $M^n \subset \mathbf{R}^{n+1}$ be a convex hypersurface. Then*

(1) *If P^2 is a two plane in $\mathbf{R}^{n+1}$ and $\Pi_{P^2}\colon \mathbf{R}^{n+1} \to P^2$ is the orthogonal projection then $L(\partial(\Pi_{P^2}(M))) \geqslant L$.*

(2) *If $P^n \subset \mathbf{R}^{n+1}$ is a hypersurface and Π_{P^n} is the orthogonal projection then* $\operatorname{Vol}(M) \geqslant 2\operatorname{Vol}(\Pi_{P^n}(M))$.

Proof of Lemma 6.2. Statement (2) is clear. Statement (1) follows from Birkhoff's idea. By slicing M with 2 planes parallel to P^2 we get a family of curves for which Birkhoff's idea works (for the details of this see the proof of Lemma 1.6 of [9]). Hence L is less than or equal to the length of the longest curve in this family. But since each curve in this family projects onto P^2, in a length preserving way, to a convex curve inside $\partial(\Pi_{P^2}(M))$, we see that the length is less than or equal to $L(\partial(\Pi_{P^2}(M)))$ and the lemma follows.

Proof of Theorem 6.1. Let K^{n+1} be the convex body such that $\partial K = M$ and let ε be the width of K, which we assume is in the direction of a line l_1. Let P^n be the hyperplane perpendicular to l_1 and $D = \Pi_{P^n}(M)$. Let w be the width of D, which we assume is in the direction of a line l_2. Let P^2 be the plane determined by l_1 and l_2. Applying Lemma 6.2 to P^2 and P^n yields:

$$L \leqslant L\big(\partial\big(\Pi_{P^2}(M)\big)\big) \leqslant 2\varepsilon + 2w \leqslant 4w$$

and

$$\operatorname{Vol}(M) \geqslant 2\operatorname{Vol}\big(\Pi_{P^n}(M)\big) \geqslant 2c_0(n)w^n.$$

Combining these two inequalities yields the theorem.

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